$\mathbb{H}$-PERFECT PSEUDO MV-ALGEBRAS AND THEIR REPRESENTATIONS

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Dedicated to Prof. Antonio Di Nola on the occasion of his 65^{th} birthday

Abstract. We study $\mathbb{H}$-perfect pseudo MV-algebras, that is, algebras which can be split into a system of ordered slices indexed by the elements of an subgroup $\mathbb{H}$ of the group of the real numbers. We show when they can be represented as a lexicographic product of $\mathbb{H}$ with some ℓ-group. In addition, we show also a categorical equivalence of this category with the category of ℓ-groups.

1. Introduction

MV-algebras were introduced by Chang in [Cha] in order to provide an algebraic counterparts of infinite-valued sentential calculus of Łukasiewicz logic. Thanks to the celebrated Representation Theorem by Mundici [Mun], such algebras are always an interval in Abelian ℓ-groups with strong unit, see also e.g. [CDM]. Recently, there appeared independently two non-commutative generalizations of MV-algebras, called pseudo MV-algebras by [GeIo] and generalized MV-algebras by [Rac], which are both equivalent. The basic result on pseudo MV-algebras from [Dvu2] says that every pseudo MV-algebra is an interval in a unital ℓ-group with strong unit which is not necessarily Abelian.

A more general structure than MV-algebras is formed by effect algebras [FoBe] which are partial algebras important for modeling quantum mechanical measurements. Such algebras are also sometimes an interval in Abelian partially ordered groups (po-groups) with strong unit. This is possible e.g. if the effect algebra has the Riesz Decomposition Property, [Rav]. For more on effect algebras, see [DvPu]. A noncommutative version of effect algebras, called pseudo MV-algebras, was presented in [DvVe1, DvVe2]. Also under a stronger type of the Riesz Decomposition Property, such algebras are intervals in po-groups with strong unit which are not

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necessarily Abelian. It is important to note that every pseudo MV-algebra can be viewed also as a pseudo effect algebra satisfying RDP_2, see [DvVe2].

We recall that a *po-group* (= partially ordered group) is a group $(G;+,0)$ (written additively) endowed with a partial order $\le$ such that if $a \le b$, $a,b \in G$, then $x+a+y \le x+b+y$ for all $x,y \in G$. We denote by $G^+ = \{g \in G : g \ge 0\}$ the *positive cone* of G. If, in addition, G is a lattice under $\le$, we call it an ℓ-group (= lattice ordered group). An element $u \in G^+$ is said to be a *strong unit* (= order unit) if given $g \in G$, there is an integer $n \ge 1$ such that $g \le nu$, and the couple (G,u) with a fixed strong unit u is said to be a *unital po-group* and a *unital ℓ-group*, respectively. For more information on po-groups and ℓ-groups and for unexplained notions, see [Fuc, Gla].

We say that an MV-algebra is perfect if every its element is either an infinitesimal or the negation of some infinitesimal. Therefore, they are mostly non Archimedean algebras. An important example of a perfect MV-algebra is the subalgebra of the Lindenbaum algebra of the first order Łukasiewicz logic generated by the class of formulas that are valid but non-provable, [DDT]. Hence, perfect MV-algebras are directly connected with the very important phenomenon of incompleteness of the Łukasiewicz first-order logic. Important results on perfect pseudo MV-algebras can be found in [DiLe1] together with their equational characterization. This notion was extended also for effect algebras in [Dvu4]. Perfect pseudo MV-algebras were studied in [Leu] and [DDT], where it was shown that such algebras are always of the form $\Gamma(\mathbb{Z} \overrightarrow{\times} G,(1,0))$, where G is an ℓ-groups. This notion was generalized for the so-called n-perfect pseudo MV-algebras, [Dvu5]. Such algebras can be split into $n+1$ comparable slices, see e.g. [DXY]. This notion was exhibited also for the case when a pseudo effect algebra can be split into a system of comparable slices indexed by the elements of a subgroup $\mathbb{H}$ of the group of real numbers $\mathbb{R}$, see [DvKo]. We note that the structure of perfect pseudo MV-algebras is very rich because there is uncountably many varieties of pseudo MV-algebras generated by the categories of perfect pseudo MV-algebras, see [DDT].

In the present paper, we study $\mathbb{H}$-perfect pseudo MV-algebras. We introduce so-called strong $\mathbb{H}$-perfect pseudo MV-algebras as algebras which can be represented as $\Gamma(\mathbb{H} \overrightarrow{\times} G,(1,0))$, where G is an ℓ-group. We present also their categorical representation by the category of ℓ-group. In addition, we introduce also weak $\mathbb{H}$-perfect pseudo MV-algebras as algebras which can be represented in the form $\Gamma(\mathbb{H} \overrightarrow{\times} G,(1,b))$, where b is a strictly positive element of an ℓ-group G.

The paper is organized as follows. Section 2 gathers elements of pseudo MV-algebras and pseudo effect algebras. Section 3 introduces $\mathbb{H}$-perfect pseudo MV-algebras. Section 4 deals with strong $\mathbb{H}$-perfect pseudo MV-algebras and it gives a representation theorem for such algebras. Section 5 shows a categorical equivalence of the category of strong $\mathbb{H}$-perfect pseudo MV-algebras with the category of ℓ-groups. Finally, Section 6 presents a representation of weak $\mathbb{H}$-perfect pseudo MV-algebras together with their categorical equivalence.

2. Pseudo MV-algebras

According to [GeIo], a *pseudo MV-algebras* (*PMV-algebra* for short) is an algebra $(M;\oplus,^-,^\sim,0,1)$ of type $(2,1,1,0,0)$ such that the following axioms hold for all $x,y,z \in M$ with an additional binary operation $\odot$ defined via

$$y \odot x = (x^- \oplus y^-)^\sim$$

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(A2) $x \oplus 0 = 0 \oplus x = x$;
(A3) $x \oplus 1 = 1 \oplus x = 1$;
(A4) $1^\sim = 0$; $1^- = 0$;
(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;
(A6) $x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$;[2]
(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;
(A8) $(x^-)^\sim = x$.

For example, if u is a strong unit of a (not necessarily Abelian) ℓ-group G,

$$\Gamma(G,u) := [0,u]$$

and

$$\begin{aligned} x \oplus y &:= (x+y) \wedge u, \\ x^- &:= u - x, \\ x^\sim &:= -x + u, \\ x \odot y &:= (x - u + y) \vee 0, \end{aligned}$$

then $(\Gamma(G,u); \oplus, ^-, ^\sim, 0, u)$ is a PMV-algebra [GeIo].

(A6) defines the join $x \vee y$ and (A7) does the meet $x \wedge y$. In addition, M with respect to $\vee$ and $\wedge$ is a distributive lattice, [GeIo].

Let $(M; \oplus, ^-, ^\sim, 0, 1)$ be a PMV-algebra. Define a partial binary operation $+$ on M via: $x + y$ is defined iff $x \le y^-$, and in this case

$$x + y := x \oplus y. \tag{2.1}$$

A PMV-algebra is an *MV-algebra* if $a \oplus b = b \oplus a$ for all $a, b \in M$. We denote by $\mathcal{PMV}$ and $\mathcal{MV}$ the variety of pseudo MV-algebras and MV-algebras, respectively.

A PMV-algebra is said to be *symmetric* if $a^- = a^\sim$ for any $a \in M$. We recall that a symmetric PMV-algebra is not necessarily an MV-algebra, see e.g. the PMV-algebra $M = \Gamma(\mathbb{H} \overrightarrow{\times} G, (1, g_0))$, where $g_0 > 0$ is not from the *commutative center* $C(G) := \{x \in G : x + y = y + x, \ \forall\, y \in G\}$. The class of all symmetric PMV-algebras forms a variety, $\mathcal{SYM}$, which contains as a proper subvariety the variety of all MV-algebras.

If A is a non-void subset of a PMV-algebra M, we set $A^- := \{a^- : a \in A\}$, $A^\sim := \{a^\sim : a \in A\}$ and if B is another non-void subset of M, we write $A \leqslant B$ if $a \le b$ for all $a \in A$ and all $b \in B$.

An *ideal* of a PMV-algebra M is any non-empty subset I of M such that (i) $a \le b \in I$ implies $a \in I$, and (ii) if $a, b \in I$, then $a \oplus b \in I$. An ideal $I \ne M$ is said to be *maximal* if it is not a proper subset of another ideal $J \ne M$; we denote by $\mathcal{M}(M)$ the set of maximal ideals of M.

According to [DvVe1, DvVe2], a partial algebraic structure $(E; +, 0, 1)$, where $+$ is a partial binary operation and 0 and 1 are constants, is called a *pseudo effect algebra* (*PEA* for short) if, for all $a, b, c \in E$, the following hold.

(PE1) $a + b$ and $(a + b) + c$ exist if and only if $b + c$ and $a + (b + c)$ exist, and in this case, $(a + b) + c = a + (b + c)$.
(PE2) There are exactly one $d \in E$ and exactly one $e \in E$ such that $a + d = e + a = 1$.
(PE3) If $a + b$ exists, there are elements $d, e \in E$ such that $a + b = d + a = b + e$.

[2] $\odot$ has a higher priority than $\oplus$.

(PE4) If $a+1$ or $1+a$ exists, then $a=0$.

If we define $a \le b$ if and only if there exists an element $c \in E$ such that $a+c=b$, then $\le$ is a partial ordering on E such that $0 \le a \le 1$ for any $a \in E$. It is possible to show that $a \le b$ if and only if $b = a+c = d+a$ for some $c, d \in E$. We write $c = a \,/\, b$ and $d = b \setminus a$. Then

$$(b \setminus a) + a = a + (a \,/\, b) = b,$$

and we write $a^- = 1 \setminus a$ and $a^\sim = a \,/\, 1$ for any $a \in E$.

If (G,u) is a unital po-group, the set $\Gamma(G,u) := \{g \in G : 0 \le g \le u\}$ endowed with the restriction of the group addition $+$ to $\Gamma(G,u)$ and $0, u$ is a pseudo effect algebra.

Let $x \in M$ and an integer $n \ge 0$ be given. We define

$$0 \odot x := 0, \quad 1 \odot x := x, \quad (n+1) \odot x := (n \odot x) \oplus x,$$
$$x^0 := 1, \quad x^1 := x, \quad x^{n+1} := x^n \odot x,$$
$$0x := 0, \quad 1x := x, \quad (n+1)x := (nx) + x,$$

if nx and $(nx)+x$ are defined in M. An element x is said to be an *infinitesimal* if mx exists in M for any integer $m \ge 1$. We denote by $\mathrm{Infinit}(M)$ the set of all infinitesimals of M.

A non-empty subset I of a PEA E is said to be an *ideal* if (i) $a, b \in I$, $a+b \in E$, then $a+b \in I$, and (ii) if $a \le b \in I$, then $a \in I$.

We introduce the following types of the Riesz Decomposition properties of po-groups:

(i) RDP if, for all $a_1, a_2, b_1, b_2 \in G^+$ such that $a_1 + a_2 = b_1 + b_2$, there are four elements $c_{11}, c_{12}, c_{21}, c_{22} \in G^+$ such that $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$ and $b_2 = c_{12} + c_{22}$;

(ii) RDP_1 if it satisfies RDP and, for the elements c_{12} and c_{21}, we have $0 \le x \le c_{12}$ and $0 \le y \le c_{21}$ imply $x+y = y+x$;

(iii) RDP_2 if it satisfies RDP and, for the elements c_{12} and c_{21}, we have $c_{12} \wedge c_{21} = 0$.

If E is a pseudo effect algebra, we say that E satisfies RDP (or RDP_1 or RDP_2) if in the later definition we change G^+ to E. Then RDP_2 implies RDP_1, and RDP_1 implies RDP; but the converse is not true, in general. A po-group G satisfies RDP_2 iff G is an ℓ-group, [DvVe1, Prop 4.2(ii)].

The basic results on PMV-algebras and PEAs are the following representation theorems [Dvu2] and [DvVe2, Thm 7.2]:

Theorem 2.1. *For any PMV-algebra $(M; \oplus, ^-, ^\sim, 0, 1)$, there exists a unique (up to isomorphism) unital ℓ-group G with a strong unit u such that $(M; \oplus, ^-, ^\sim, 0, 1) \cong (\Gamma(G,u); \oplus, ^-, ^\sim, 0, u)$. The functor Γ defines a categorical equivalence of the variety of PMV-algebras with the category of unital ℓ-groups.*

Theorem 2.2. *For every PEA $(E; +, 0, 1)$ with RDP_1, there is a unique unital po-group (G,u) with RDP_1 such that $(E; +, 0, 1) \cong (\Gamma(G,u); +, 0, u)$. The functor Γ defines a categorical equivalence of the category of PEAs with the category of unital po-groups with RDP_1.*

In [DvVe2, Thm 8.3, 8.4], it was proved that if $(M; \oplus, ^-, ^\sim, 0, 1)$ is a PMV-algebra, then $(M; +, 0, 1)$, where $+$ is defined by (2.1), is a pseudo effect algebra

with RDP_2. Conversely, if $(E;+,0,1)$ is a pseudo effect algebra with RDP_2, then E is a lattice, and by [DvVe2, Thm 8.8], $(E;\oplus,^-,^\sim,0,1)$, where

$$a \oplus b := (b^- \setminus (a \wedge b^-))^\sim, \tag{2.2}$$

is a PMV-algebra. In addition, a PEA E has RDP_2 iff E is a lattice and E satisfies RDP_1, see [DvVe2, Thm 8.8].

We note that if M is a PMV-algebra, then the notion of an ideal of an PMV-algebra M coincides with the notion of an ideal taken in the PEA M with $+$ defined by (2.1).

Let A and B be two non-void subsets of a PMV-algebra M, we set (i) $A\oplus B := \{a\oplus b : a\in A, b\in B\}$, (ii) $A+B = \{a+b : \text{if } a+b \text{ exists in } M \text{ for } a\in A,\ b\in B\}$. We say that $A+B$ is *defined* in M if $a+b$ exists in M for any $a\in A$ and any $b\in B$.

An ideal I of M is normal if $x\oplus I = I\oplus x$ for any $x\in M$; let $\mathcal{N}(M)$ be the set of normal ideals of M. There is a one-to-one correspondence between normal ideals and congruences for PMV-algebras, [GeIo, Thm 3.8]. The quotient PMV-algebra over a normal ideal I, M/I, is defined as the set of all elements of the form $x/I := \{y\in M : x\odot y^- \oplus y\odot x^- \in I\}$, or equivalently, $x/I := \{y\in M : x^\sim \odot y \oplus y^\sim \odot x \in I\}$.

We can define a maximal ideal of a PEA E in the same way as for PMV-algebras, and an ideal I of M is *normal* if $x+I = I+x$ for any $x\in M$. We note the normality of an ideal of an PMV-algebra M is the same as that for the PEA M with $+$ determined by (2.1).

We define (i) the *radical* of a PMV-algebra M, $\mathrm{Rad}(M)$, as the set

$$\mathrm{Rad}(M) = \bigcap\{I :\ I\in\mathcal{M}(M)\},$$

and (ii) the *normal radical* of M, $\mathrm{Rad}_n(M)$, via

$$\mathrm{Rad}_n(M) = \bigcap\{I :\ I\in\mathcal{N}(M)\cap\mathcal{M}(M)\}.$$

By [DDJ, Prop. 4.1, Thm 4.2], it is possible to show that

$$\mathrm{Rad}(M) \subseteq \mathrm{Infinit}(M) \subseteq \mathrm{Rad}_n(M). \tag{2.3}$$

Finally, we say that a mapping $s : M \to [0,1]$ is a *state* on a PMV-algebra M if (i) $s(1)=1$, and (ii) $s(a+b) = s(a)+s(b)$ whenever $a+b$ is defined in M. We say that a state s is *extremal* if from $s = \lambda s_1 + (1-\lambda)s_2$, where s_1, s_2 are states on M and λ is a real number such that $0<\lambda<1$, it follows $s = s_1 = s_2$. We denote by $\mathcal{S}(M)$ and $\partial_e\mathcal{S}(M)$ the set of all states and the set of all extremal states on M, respectively. If M is an MV-algebra, $\mathcal{S}(M)$ is always a non-void set. But if M is a PMV-algebra, it can happen that M is stateless, see e.g. [DDJ, Dvu1, DvHo]. The set $\mathrm{Ker}(s) = \{a\in M : s(a)=0\}$, the *kernel* of s, is a normal ideal. A state s is extremal iff $\mathrm{Ker}(s)$ is a maximal ideal, and conversely, every maximal and normal ideal is a kernel of a unique extremal state, see [Dvu1]. In addition, a state s is extremal iff $s(a\wedge b) = \min\{s(a), s(b)\}$, $a,b\in M$, [Dvu1, Prop 4.7].

A state on a unital ℓ-group (G,u) is a mapping $s : G\to\mathbb{R}$ such that (i) $s(G^+)\subseteq \mathbb{R}^+$, (ii) $s(g_1+g_2) = s(g_1)+s(g_2)$ for all $g_1, g_2\in G$, and (iii) $s(u)=1$. There is a one-to-one correspondence between the states on (G,u) and $\Gamma(G,u)$; every state on $\Gamma(G,u)$ can be extended to a unique state on (G,u), see [Dvu1].

3. $\mathbb{H}$-perfect PMV-algebras

From this section, $\mathbb{H}$ will denote a subgroup of the group of real numbers $\mathbb{R}$ such that $1 \in \mathbb{H}$. The main aim of this section is to introduce and study PMV-algebras which can be split into a family of comparable slices indexed by the elements of the subgroup $\mathbb{H}$. Such prototypical examples are PMV-algebras represented in the form

$$\Gamma(\mathbb{H} \overrightarrow{\times} G, (1,0)), \tag{3.1}$$

where G is any ℓ-group, and $\mathbb{H} \overrightarrow{\times} G$ denotes the *lexicographic product* of $\mathbb{H}$ with G; we note that in such a lexicographic product, the order $\le$ is defined as follows: $(h_1, g_1) \le (h_2, g_2)$ iff either $h_1 < h_2$ or $h_1 = h_2$ and $g_1 \le g_2$. It is clear that the element $u = (1,0)$ is a strong unit for $\mathbb{H} \overrightarrow{\times} G$ and (3.1) defines a PMV-algebra.

A very special case is when $G = O$, where O is the zero ℓ-group, because then $\Gamma(\mathbb{H} \overrightarrow{\times} O, (1,0))$ is isomorphic to the Archimedean MV-algebra $\Gamma(\mathbb{Z}, 1)$. In general, if $G \ne O$, (3.1) does not give an Archimedean PMV-algebra.

By $\mathbb{Q}$ we denote the group of rational numbers in $\mathbb{R}$, $\mathbb{Z}$ denotes the group of integers, and given an integer $n \ge 1$, $\frac{1}{n}\mathbb{Z} := \{\frac{i}{n} : i \in \mathbb{Z}\}$. By [Go, Lem 4.21], every $\mathbb{H}$ is either *cyclic*, i.e. $\mathbb{H} = \frac{1}{n}\mathbb{Z}$ for some $n \ge 1$ or $\mathbb{H}$ is dense in $\mathbb{R}$.

For example, if $\mathbb{H} = \mathbb{H}(\alpha)$ is a subgroup of $\mathbb{R}$ generated by $\alpha \in [0,1]$ and 1, then $\mathbb{H} = \frac{1}{n}\mathbb{Z}$ for some integer $n \ge 1$ if α is a rational number. Otherwise, $\mathbb{H}(\alpha)$ is countable and dense in $\mathbb{R}$, and $M(\alpha) := \Gamma(\mathbb{H}(\alpha), 1) = \{m + n\alpha : m, n \in \mathbb{Z},\ 0 \le m + n\alpha \le 1\}$, see [CDM, p. 149]. In addition, $\{\mathbb{H}(\alpha) : \alpha \in (0,1)\}$ is an uncountable system of non-isomorphic subgroups of $\mathbb{R}$.

We set $[0,1]_{\mathbb{H}} := [0,1] \cap \mathbb{H}$.

Definition 3.1. We say that a PMV-algebra M is $\mathbb{H}$-*perfect*, if there is a system $(M_t : t \in [0,1]_{\mathbb{H}})$ of nonempty subsets of M such that it is an $\mathbb{H}$-*decomposition* of M, i.e. $M_s \cap M_t = \emptyset$ for $s < t$, $s, t \in [0,1]_{\mathbb{H}}$ and $\bigcup_{t \in [0,1]_{\mathbb{H}}} = M$ and

- (a) $M_s \leqslant M_t$ for all $s < t$, $s, t \in [0,1]_{\mathbb{H}}$,
- (b) $M_t^- = M_{1-t} = M_t^\sim$ for any $t \in [0,1]_{\mathbb{H}}$.
- (c) if $x \in M_v$ and $y \in M_t$, then $x \oplus y \in M_{v \oplus t}$, where $v \oplus t = \min\{v + t, 1\}$.

We recall that if $\mathbb{H} = \frac{1}{n}\mathbb{Z}$, a $\frac{1}{n}\mathbb{Z}$-perfect PMV-algebra is said to be *n-perfect*, for more details on n-perfect PMV-algebras, see [Dvu5].

For example, let $M = \Gamma(\mathbb{H} \overrightarrow{\times} G, (1,0))$. We set $M_0 = \{(0,g) : g \in G^+\}$, $M_1 := \{(1,-g) : g \in G^+\}$ and for $t \in [0,1]_{\mathbb{H}} \setminus \{0,1\}$, we define $M_t := \{(t,g) : g \in G\}$. Then $(M_t : t \in [0,1]_{\mathbb{H}})$ is an $\mathbb{H}$-decomposition of M and M is an $\mathbb{H}$-perfect PMV-algebra.

Sometimes we will write also $M = (M_t : t \in [0,1]_{\mathbb{H}})$ for $\mathbb{H}$-perfect PMV-algebras.

We say that a state s on a PMV-algebra M is an $\mathbb{H}$-*valued state* if $s(M) = \mathbb{H}$. If $s(M) \subseteq [0,1]_{\mathbb{H}}$, we say that s is an $\mathbb{H}$-*state*. In particular, if $\mathbb{H} = \frac{1}{n}\mathbb{Z}$, a $\frac{1}{n}\mathbb{Z}$-valued state is also said to be an $(n+1)$-*valued discrete state*, [DXY].

The basic properties of $\mathbb{H}$-perfect PMV-algebras are described as follows.

Theorem 3.2. *Let $M = (M_t : t \in [0,1]_{\mathbb{H}})$ be an $\mathbb{H}$-perfect PMV-algebra.*

- (i) *Let $a \in M_v$, $b \in M_t$. If $v+t < 1$, then $a+b$ is defined in M and $a+b \in M_{v+t}$; if $a+b$ is defined in M, then $v+t \le 1$.*
- (ii) *$M_v + M_t$ is defined in M and $M_v + M_t = M_{v+t}$ whenever $v+t < 1$.*
- (iii) *If $a \in M_v$ and $b \in M_t$, and $v+t > 1$, then $a+b$ is not defined in M.*

(iv) *M admits a unique state. This state is an $\mathbb{H}$-valued state such that $s(M_t) = \{t\}$ for each $t \in [0,1]_{\mathbb{H}}$. Then $M_t = s^{-1}(\{t\})$ for any $t \in [0,1]_{\mathbb{H}}$, and s is an $\mathbb{H}$-valued state such that* $\mathrm{Ker}(s) = M_0$.
(v) *M_0 is a normal and maximal ideal of M such that $M_0 + M_0 = M_0$.*
(vi) *M_0 is a unique maximal ideal of M, and* $M_0 = \mathrm{Rad}(M) = \mathrm{Infinit}(M)$.
(vii) *Let $M = (M'_t : t \in [0,1]_{\mathbb{H}})$ be another representation of M satisfying* (a)–(c) *of Definition* 3.1*, then $M_t = M'_t$ for each $t \in [0,1]_{\mathbb{H}}$.*
(viii) *The quotient PMV-algebra $M/M_0 \cong \Gamma(\mathbb{H}, 1)$.*

Proof. (i) Assume $a \in M_v$ and $b \in M_t$ for $v + t < 1$. Then $b^- \in M_{1-t}$, so that $a \leq b^-$, and $a + b$ is defined in M. Conversely, let $a + b$ be defined, then $a \leq b^- \in M_{1-t}$ which gives $v + t \leq 1$.

(ii) By (i), we have $M_v + M_t \subseteq M_{v+t}$. Suppose $z \in M_{v+t}$. Then, for any $x \in M_v$, we have $x \leq z$, and hence $y = z \setminus x$ is defined in M, and $y \in M_w$ for some $w \in [0,1]_{\mathbb{H}}$. Since $z = y + x \in M_{v+t} \cap M_{v+w}$, we conclude $t = w$ and $M_{v+t} \subseteq M_v + M_t$.

(iii) If $a + b \in M$, then $a \leq b^- \in M_{1-t} \leqslant M_v$ which gives $a \leq b^- \leq a$, that is, $a = b^-$. This is possible only if $v = 1 - t$ which is impossible.

(iv)–(vi) Define a mapping $s:\ M \to [0,1]$ by $s(x) = t$ if $x \in M_t$. It is clear that s is a well-defined mapping. Take $a, b \in M$ such that $a + b$ is defined in M. Then there are unique indices v and t such that $a \in M_v$ and $b \in M_t$. By (i), $v + t \leq 1$ and $a + b \in M_{v+t}$. Therefore, $s(a + b) = v + t = s(a) + s(b)$. It is evident that $s(1) = 1$, $\mathrm{Ker}(s) = M_0$, and $M_t = s^{-1}(\{t\})$ for $t \in [0,1]_{\mathbb{H}}$. In particular, M_0 is a normal ideal of M.

Maximality of M_0. Take $x \in M_t \setminus M_0$, where $0 < t < 1$, $t \in [0,1]_{\mathbb{H}}$. Let I be an ideal of M generated by M_0 and x. Then, for every $v < t$, $s \in [0,1]_{\mathbb{H}}$, we have $M_v \leqslant M_t$, whence $M_v \subseteq I$. There are two cases: (a) there is no $v \in [0,1]_{\mathbb{H}}$ such that $0 < v < t$. Then $t = 1/n$ for some integer $n \geq 1$ and $\mathbb{H} = \frac{1}{n}\mathbb{Z}$. If $n = 1$, then $s(x) = 1$, $s(x^-) = 0$, and $x^- \in M_0$. Hence, $1 \in I$.

If $n \geq 2$, then $y := (n-1)x$ is defined in M, and $y \in I$. For the element y^-, we have $s(y^-) = 1/n$, so that $y^- \in I$ which means $1 \in I$.

(b) $\mathbb{H}$ is no cyclic subgroup of $\mathbb{R}$, so that it is dense in $\mathbb{R}$. There is a strictly decreasing sequence $\{t_i\}$ of non-zero elements of $[0,1]_{\mathbb{H}}$ such that $t_i \searrow 0$. For every t_i, there is a maximal integer m_i such that $y_i := m_i t_i$ is defined in M. Hence, for enough small t_i, $s(y_i^-) < t$ so that $y_i^- \in I$ which again proves $I = M$, and M_0 is a maximal ideal.

Uniqueness of a maximal ideal. Assume that I is another maximal ideal of M. Let there be $x \in M_t \cap I$ for some $t \in [0,1]_{\mathbb{H}}$, $t > 0$. Then, for every $z \in M_0$, we have $z \leq x$ and $z \in I$, so that $M_0 \subseteq I$. The maximality of M_0 yields $M_0 = I$.

Since there is a one-to-one correspondence between extremal states and maximal ideals which are also normal given by $s \leftrightarrow \mathrm{Ker}(s)$, [Dvu1, Prop 4.3-4.6], we see that M has a unique state, this state is extremal and an $\mathbb{H}$-valued state.

Finally, we show $M_0 = \mathrm{Infinit}(M)$. Since $M_0 + M_0 = M_0$, we have $M_0 \subseteq \mathrm{Infinit}(M)$. Let $x \in \mathrm{Infinit}(M)$. Then mx exists in M for any integer $m \geq 1$. Hence, $s(mx) = ms(x) \leq 1$ which gives $s(x) = 0$ and $x \in \mathrm{Ker}(s) = M_0$. From (2.3), we conclude $M_0 = \mathrm{Rad}(M) = \mathrm{Infinit}(M)$.

(vii) If $M = (M'_t : t \in [0,1]_{\mathbb{H}})$ is another representation of M, then by (iv), M admits a state s' such that $M'_t = s'^{-1}(\{t\})$ for any $t \in [0,1]_{\mathbb{H}}$. Since M admits a unique state, $s = s'$ and $M_t = M'_t$ for each $t \in [0,1]_{\mathbb{H}}$.

(viii) By (iv), there is a (unique extremal) state s on M such that $\mathrm{Ker}(s) = M_0$. Then $a \sim b$ iff $s(a) = s(a \wedge b) = s(b)$. Since s is an extremal state, $\mathrm{Ker}(s)$ is a maximal ideal and normal. Hence, $M/\mathrm{Ker}(s) = [0,1]_{\mathbb{H}} = \Gamma(\mathbb{H},1)$. Hence, $M/M_0 \cong \Gamma(\mathbb{H},1)$. $\square$

In the rest of this section, we will study some varieties of PMV-algebras generated by $\mathbb{H}$-perfect PMV-algebras. We show that there are two important cases depending on whether $\mathbb{H}$ is a cyclic or non-cyclic subgroup of $\mathbb{R}$. We note that the cyclic case was studied in [Dvu6].

If $\mathcal{K}$ is a family of PMV-algebras, we denote by $\mathcal{V}(\mathcal{K})$ the variety of PMV-algebras generated by $\mathcal{K}$. If $\mathcal{K} = \{K\}$, we denote simply $\mathcal{V}(K) := \mathcal{V}(\mathcal{K})$.

To show these varieties, we introduce so-called top varieties of PMV-algebras, see [DvHo]. The basic tool in our considerations is Theorem 2.1. In particular, it entails a one-to-one correspondence between the set of ideals, normal ideals, maximal ideals of $M = \Gamma(G,u)$, and the set of convex ℓ-subgroups, $\mathcal{C}(G)$, ℓ-ideals, $\mathcal{L}(G)$, and maximal convex ℓ-subgroups, $\mathcal{M}(G)$, of (G,u), see [Dvu1]; the one-to-one mapping $\psi:\ \mathcal{I}(M) \to \mathcal{C}(G)$ is defined by

$$\psi(I) = \{x \in G:\ \exists\, x_i, y_j \in I,\quad x = x_1 + \cdots + x_n - y_1 - \cdots - y_m\}. \tag{3.2}$$

Let $M = \Gamma(G,u)$ be a PMV-algebra, where (G,u) is a unital ℓ-group. By a *value* of u in (G,u) we mean a convex ℓ-subgroup H of (G,u) maximal under condition H does not contain u. Hence, $\psi^{-1}(H)$ is a maximal ideal of M, where ψ is defined by (3.2), and vice versa. If I is a maximal ideal of M, then $\psi(I)$ is a value of u in (G,u).

For any value V of (G,u), we set

$$K(V) = \bigcap_{g \in G} g^{-1} V g$$

(for a moment we use a multiplicative form of (G,u)). Then $K(V)$ is a normal convex ℓ-subgroup of (G,u) contained in V, and $(G/K(V), G/V)$ is a primitive transitive ℓ-permutation group called a *top component* of G.

Let $\mathcal{V}$ be a variety of PMV-algebras and let $\Gamma^{-1}(\mathcal{V}) = \{(G,u):\ \Gamma(G,u) \in \mathcal{V}\}$. We recall that $\mathcal{V}$ contains a trivial PMV-algebra (i.e. $0 = 1$). Then by [DvHo, Thm 3.1], $\Gamma^{-1}(\mathcal{V})$ is an equational class of unital ℓ-groups in some extended sense: $\Gamma^{-1}(\mathcal{V})$ is not a variety in the usual sense of universal algebra, but rather a class of unital ℓ-groups described by equations in the language of unital ℓ-groups.

Let

$$\mathcal{T}(\mathcal{V}) = \{\Gamma(G,u):\ \Gamma(G/K(V), u/K(V)) \in \mathcal{V},\ V \in \mathcal{M}(G)\} \cup \{\{0\}\}. \tag{3.3}$$

By [DvHo, Cor. 4.3], $\mathcal{T}(\mathcal{V})$ is a variety, we call it a *top variety* of $\mathcal{V}$.

We denote by $\mathcal{M}$ the set of PMV-algebras M such that either every maximal ideal of M is normal or M is trivial. In [DDT, (6.1)], there was shown that $\mathcal{M}$ is a variety such that

$$\mathcal{M} = \mathcal{T}(\mathcal{MV}) = \mathcal{T}(\mathcal{N}) = \mathcal{T}(\mathcal{M}), \tag{3.4}$$

where $\mathcal{MV}$, as it was already mentioned, is the variety of MV-algebras and $\mathcal{N}$ is the set of normal-valued PMV-algebras, which according to [Dvu1, Thm 6.8] is a variety. (We recall that a *value* of any non-zero element $b \in M$ is any ideal I of M maximal under the condition $b \notin I$. The ideal I^* generated by I and b is said to be a *cover* of I and we say that I is normal in its cover if $x \oplus I = I \oplus x$ for any $x \in I^*$. Finally, we say that M is *normal-valued* if every value is normal in its cover.)

We recall that according to Theorem 2.1, it is possible to show that a PMV-algebra $M = \Gamma(G, u)$ is symmetric iff $u \in C(G)$, [Dvu3, p. 98].

Let $\mathbb{H}$ be a subgroup of $\mathbb{R}$ such that $1 \in \mathbb{H}$. We define $\mathcal{PPMV}_{\mathbb{H}}$, the system of $\mathbb{H}$-perfect PMV-algebras ($\mathcal{PPMV}_{\mathbb{H}}^S$ symmetric $\mathbb{H}$-perfect PMV-algebras), $\mathcal{V}(\mathcal{PPMV}_{\mathbb{H}})$, the variety generated by all $\mathbb{H}$-perfect PMV-algebras, and $\mathcal{BP}_{\mathbb{H}}$ (and $\mathcal{SBP}_{\mathbb{H}}$), the system of (symmetric) PMV-algebras M such that either every maximal ideal of M is normal and every extremal state of M an $\mathbb{H}$-state or M is the one-element PMV-algebra. Or equivalently, either every maximal ideal I of M is normal and M/I is a subalgebra of $\Gamma(\mathbb{H}, 1)$.

If $\mathbb{H} = \frac{1}{n}\mathbb{Z}$, instead of $\mathcal{PPMV}_{\mathbb{H}}$, $\mathcal{BP}_{\mathbb{H}}$ and $\mathcal{SBP}_{\mathbb{H}}$, we write according to [Dvu5], $\mathcal{PPMV}_n$, $\mathcal{BP}_n$ and $\mathcal{SBP}_n$, respectively.

In such a case, $\mathcal{BP}_n$ consists of all PMV-algebras M such that every maximal ideal is normal and every extremal state is $(k+1)$-valued, where k divides n, or M is the one-element PMV-algebra. Or equivalently, either every maximal ideal I of M is normal and $M/I \cong \Gamma(\mathbb{Z}, k)$ where $k|n$, or $M = \{0\}$. It is clear that $\mathcal{BP}_1 = \mathcal{BP}$, and $\mathcal{SBP}_1 = \mathcal{SBP}$, where $\mathcal{BP}$ and $\mathcal{SBP}$ were studied in [DDT]. We have $\mathcal{BP}_m \subseteq \mathcal{BP}_n$ iff $m|n$. If n is prime, then $\mathcal{BP}_n$ is of particular interest.

In [DiLe2, Cor. 11], there is presented a characterization of MV-algebras which are members of the variety $\mathcal{V}(\mathcal{M}_n(\mathbb{Z}))$, that is, the variety generated by the MV-algebra $\Gamma(\frac{1}{n}\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (1,0))$. They showed that the variety $\mathcal{V}(\mathcal{M}_n(\mathbb{Z}))$ is characterized by the following identities

$$((n+1) \odot x^n)^2 = 2 \odot x^{n+1}, \tag{3.5}$$

$$(p \odot x^{p-1})^{n+1} = (n+1) \odot x^p, \tag{3.6}$$

for every integer p, $1 < p < n$, such that p is not a divisor of n.

These identities were used to describe the following varieties. Let $\mathcal{V}_{P_n}$ and $\mathcal{V}_{P_n}^S$ be the varieties of PMV-algebras and symmetric PMV-algebras, respectively, satisfying the identities (3.5)–(3.6). Then the following result was established in [Dvu5, Thm 5.1].

Theorem 3.3. *We have $\mathcal{T}(\mathcal{V}_{P_n}) = \mathcal{BP}_n$, and $\mathcal{BP}_n$ is a variety such that $\mathcal{T}(\mathcal{BP}_n) = \mathcal{BP}_n = \mathcal{T}(\mathcal{V}(\Gamma(\mathbb{Z}, n))) = \mathcal{T}(\mathcal{V}(\mathcal{M}_n(\mathbb{Z})))$.*

For the case that $\mathbb{H}$ is not cyclic, we extend Theorem 3.3 as follows. We note that by (3.3) we can define $\mathcal{T}(\mathcal{V})$ for any family $\mathcal{V}$ of PMV-algebras (not only for varieties).

Theorem 3.4. *Let $\mathbb{H}$ be not a cyclic subgroup of $\mathbb{R}$. Then $\mathcal{T}(\mathcal{BP}_{\mathbb{H}}) = \mathcal{BP}_{\mathbb{H}}$ and $\mathcal{BP}_{\mathbb{H}}$. In addition, $\mathcal{T}(\mathcal{V}(\mathcal{BP}_{\mathbb{H}})) = \mathcal{M} = \mathcal{T}(\mathcal{V}(\mathcal{PPMV}_{\mathbb{H}}))$.*

Proof. By the definition of $\mathcal{BP}_{\mathbb{H}}$, we have $\mathcal{BP}_{\mathbb{H}} \subset \mathcal{M}$. Due to (3.4), $\mathcal{BP}_{\mathbb{H}} \subseteq \mathcal{T}(\mathcal{BP}_{\mathbb{H}}) \subseteq \mathcal{M}$. Let $M \in \mathcal{T}(\mathcal{BP}_{\mathbb{H}})$ and let I be a maximal ideal of M. Then I is normal and $M/I \in \mathcal{BP}_{\mathbb{H}}$. Since I is maximal, M/I is an MV-subalgebra of

$\Gamma(\mathbb{H},1)\subseteq\Gamma(\mathbb{R},1)$ and M/I has a unique maximal ideal, J, which is the zero one. Therefore, $M/I\cong (M/I)/J\in\mathcal{BP}_\mathbb{H}$. This proves that $\mathcal{BP}_\mathbb{H}=\mathcal{T}(\mathcal{BP}_\mathbb{H})$.

By (iv) of Theorem 3.2, we have $\mathcal{PPMV}_\mathbb{H}\subseteq\mathcal{BP}_\mathbb{H}\subseteq\mathcal{M}$. Then $\mathcal{V}(\mathcal{PPMV}_\mathbb{H})\subseteq\mathcal{V}(\mathcal{BP}_\mathbb{H})\subseteq\mathcal{M}$. It is clear that $\Gamma(\mathbb{H},1)\in\mathcal{V}(\mathcal{PPMV}_\mathbb{H})$. Since H is dense in $\mathbb{R}$, by [CDM, Prop 8.1.1], $\mathcal{MV}=\mathcal{V}(\Gamma(\mathbb{H},1))$ and, therefore by (3.4), $\mathcal{M}=\mathcal{T}(\mathcal{MV})\subseteq\mathcal{T}(\mathcal{V}(\mathcal{PPMV}_\mathbb{H}))\subseteq\mathcal{T}(\mathcal{V}(\mathcal{BP}_\mathbb{H}))\subseteq\mathcal{T}(\mathcal{M})=\mathcal{M}$. $\square$

We note that according to Theorem 3.3, if $\mathbb{H}$ is cyclic, then $\mathcal{BP}_\mathbb{H}$ is a variety. In the next theorem, we show that if $\mathbb{H}\neq\mathbb{R}$ is not cyclic, then $\mathcal{BP}_\mathbb{H}$ is not a variety.

Now we show when $\mathcal{BP}_\mathbb{H}$ is a variety.

Theorem 3.5. *The systems $\mathcal{BP}_\mathbb{H}$ and $\mathcal{SBP}_\mathbb{H}$ are varieties if and only if either $\mathbb{H}$ is cyclic or $\mathbb{H}=\mathbb{R}$. In such a case, $\mathcal{BP}_\mathbb{R}=\mathcal{M}$, $\mathcal{SBP}_\mathbb{R}=\mathcal{SYM}\cap\mathcal{M}$, and all $\mathcal{BP}_n\neq\mathcal{M}$ are mutually different.*

Proof. The case when $\mathcal{BP}_\mathbb{H}$ is a variety for $\mathbb{H}=\frac{1}{n}\mathbb{Z}$ was shown in Theorem 3.3. If $\mathbb{H}=\mathbb{R}$, then evidently $\mathcal{BP}_\mathbb{R}\subseteq\mathcal{M}$ and if $M\in\mathcal{M}$, then every its maximal ideal I is normal, and M/I is a subalgebra of $\Gamma(\mathbb{R},1)$, so that $M\in\mathcal{BP}_\mathbb{R}$.

Now assume that $\mathbb{H}$ is not a cyclic subgroup of $\mathbb{R}$ and let $\mathbb{H}\neq\mathbb{R}$. If $\mathcal{BP}_\mathbb{H}$ is a variety, by Theorem 3.4, $\mathcal{BP}_\mathbb{H}=\mathcal{T}(\mathcal{BP}_\mathbb{H})$ and $\mathcal{MV}\subseteq\mathcal{BP}_\mathbb{H}$ so that $M=\Gamma(\mathbb{R},1)\in\mathcal{MV}\subseteq\mathcal{BP}_\mathbb{H}$, but on the other hand, M does not belong to $\mathcal{BP}_\mathbb{H}$ by definition of $\mathcal{BP}_\mathbb{H}$ because $\mathbb{R}$ is not a subgroup of $\mathbb{H}$.

In a similar way we deal with $\mathcal{SBP}_\mathbb{H}$. $\square$

In what follows, we describe subdirectly irreducible elements in $\mathcal{BP}_\mathbb{H}$, Theorem 3.7. It will be shown that they are only $\mathbb{K}$-perfect PMV-algebras, where $\mathbb{K}$ is a subgroup of $\mathbb{H}$ such that $1\in\mathbb{K}$.

If A is a subset of a PMV-algebra M, we denote by $\langle A\rangle$ the subalgebra of M generated by A.

Proposition 3.6. (1) *Let M be a PMV-algebra such that $\mathcal{S}(M)\neq\emptyset$, and let us define*

$$M'_t=\bigcap\{s^{-1}(\{t\}):\ s\in\partial_e\mathcal{S}(M)\},\quad t\in[0,1]_\mathbb{H}.$$

Then

$$\langle\bigcup_{t\in[0,1]_\mathbb{H}}M'_t\rangle=\bigcup_{t\in[0,1]_\mathbb{H}}M'_t.$$

(2) *If $M\in\mathcal{M}$, then $\bigcup_{t\in[0,1]_\mathbb{H}}M'_t$ is the biggest subalgebra of M having a unique extremal state, and this state is an $\mathbb{H}$-state.*

Proof. (1) It is clear that $M':=\bigcup_{t\in[0,1]_\mathbb{H}}M'_t$ contains $0,1$, and if $x\in M'_t$, then $x^-,x^\sim\in M'_{1-t}$, [Dvu1, Prop. 4.1]. If $x\in M'_v$ and $y\in M'_t$, then $x\oplus y\in M'_{v\oplus t}$.

(2) If s_1 and s_2 are extremal states on M, then their restrictions to M' are extremal states on M' which are $\mathbb{H}$-states, and $s_1(a)=s_2(a)$ for any $a\in M'$. Conversely, if s is an extremal state on M', then there is an extremal state $\hat{s}$ on M such that $\mathrm{Ker}(s)=\mathrm{Ker}(\hat{s})\cap M'$. Then $\mathrm{Ker}(s)=\mathrm{Ker}(s_{|M'})$ which yields $s=\hat{s}_{|M'}$. Therefore, $s=s_{1|M'}$ for any extremal state s_1 on M. Let s' be the unique extremal state on M', then $M'_t=s'^{-1}(\{t\})$ whenever $M'_t\neq\emptyset$ for any $t\in[0,1]_\mathbb{H}$.

Let now M'' be an arbitrary subalgebra of M having a unique extremal state s'', and let this state be an $\mathbb{H}$-state. Since every restriction of an extremal state of M to M'' is an extremal state on M'', and any extremal state on M'' can be extended

to an extremal state on M, we see that $s''^{-1}(\{t\}) \subseteq M'_t$ for any $t \in [0,1]_{\mathbb{H}}$, hence, $M'' \subseteq M'$. $\square$

The following characterization of subdirectly irreducible elements was originally proved in [Dvu5, Lem 5.3] for the case $\mathbb{H} = \frac{1}{n}\mathbb{Z}$. In the following lemma we extend it for a general case of $\mathbb{H}$. Nevertheless the proof for our case follows the same ideas as that in [Dvu5], to be self-contained, we present the proof if full completeness together with necessary changes.

Theorem 3.7. *If $M \in \mathcal{BP}_{\mathbb{H}}$ ($M \in \mathcal{SBP}_{\mathbb{H}}$) is subdirectly irreducible, then either M is trivial or $M = \bigcup_{t\in[0,1]_{\mathbb{H}}} M_t$, where $M_t = \bigcap\{s^{-1}(\{t\}) : \ s \in \partial_e\mathcal{S}(M)\}$ for each $t \in [0,1]_{\mathbb{H}}$, $|\partial_e\mathcal{S}(M)| = 1$, and M is a $\mathbb{K}$-perfect PMV-algebra (symmetric and $\mathbb{K}$-perfect PMV-algebra), where $\mathbb{K}$ is a subgroup of $\mathbb{H}$ such that $1 \in \mathbb{K}$.*

Proof. Assume $M = \Gamma(G,u)$ for a unital ℓ-group (G,u) is non-trivial. Due to Theorem 2.1, M is subdirectly irreducible iff G is subdirectly irreducible. In view of [Gla, Cor. 7.1.3], G has a faithful transitive representation. Therefore, by [Gla, Cor. 7.1.1], this is possible iff there is a prime subgroup C of G such that $\bigcap_{g\in G} g^{-1}Cg = \{1\}$ (we use the multiplicative form of (G,u)). In such a case, the set $\Omega := \{Cg : g \in G\}$ of right cosets of C is totally ordered assuming $Cg \le Ch$ iff $g \le ch$ for some $c \in C$, and G has a faithful transitive representation on Ω, namely $\psi(f) = Cgf$, $f \in G$, with $\mathrm{Ker}(\psi) = \bigcap_{g\in G} g^{-1}Cg = \{1\}$.

Since the system of prime subgroups of G forms a root system, there is a unique maximal ideal I of M such that $C \subseteq \psi(I) =: \hat{I}$, where $\psi(I)$ is defined by (3.2).

(I) Assume $M/I \cong \Gamma(\mathbb{Z}, n)$. Due to the one-to-one correspondence between normal and maximal ideals, I, and extremal states, s, given by $I = \mathrm{Ker}(s)$, let the maximal ideal I correspond to a unique extremal state, say s_I. We define $I_t = s_I^{-1}(\{t\})$ for any $t \in [0,1]_{\mathbb{H}}$. Then $M = \bigcup_{t\in[0,1]_{\mathbb{H}}} I_t$.

Claim 1. If $a \in I$ and $b \notin I$, then $a \le b$.

There are two possibilities: (1) $Cg = Cg(a \wedge b)$ and (2) $Cg \ne Cg(a \wedge b)$.

(1) Let $Cg = Cg(a\wedge b)$. Then $a\wedge b \in g^{-1}Cg \subseteq g^{-1}\hat{I}g = \hat{I}$. Because $g^{-1}Cg$ is also prime, we have $a \in g^{-1}Cg$. Hence, $Cga = Cg$, i.e., $Cga = Cg = Cg(a \wedge b) \le Cgb$.

(2) Let $Cg \ne Cg(a\wedge b)$. The transitivity of G entails there is an $h \in G$ such that $Cgh = Cg(a \wedge b)$. Then $gh = cg(a \wedge b)$ for some $c \in C$, and $h = g^{-1}cg(a \wedge b) \in \hat{I}$. Hence, $Cgh = Cghh^{-1}(a \wedge b)$ and $h^{-1}(a \wedge b) = (h^{-1}a) \wedge (h^{-1}b) \in (gh)^{-1}C(gh)$. Since $(gh)^{-1}C(gh)$ is prime, and $h \in \hat{I}$, we get $h^{-1}a \in (gh)^{-1}C(gh)$. Then $h^{-1}a = (gh)^{-1}cgh$ for some $c \in C$, and $ga = ghh^{-1}a = cgh$, i.e., $Cga = Cgh$. But $Cga = Cgh = Cg(a \wedge b) \le Cgb$.

Combining (1) and (2), we get $Cga \le Cgb$ for any $g \in G$, i.e., $a \le x \wedge b \le x$, and $a = a \wedge b$ proving Claim 1.

Claim 2. If s is an arbitrary extremal state on M, $s(x) = s_I(x)$ for any $x \in I$.

Let $x \in I = \mathrm{Ker}(s_I)$, then by Claim 1, $x \le x^-$ and $k \odot x \le (k \odot x)^-$ for any integer $k \ge 1$. We assert that $s(x) = 0$. If not, then $s(x) = t$ for some $t \in [0,1]_{\mathbb{H}}$. Hence, $1 = s(n\odot x) \le s((n\odot x)^-) = 0$ which is a contradiction. Therefore, $s(x) = 0$. Hence $\mathrm{Ker}(s_I) \subseteq \mathrm{Ker}(s)$. Since s_I and s are extremal, their kernels are maximal ideals, so that, $\mathrm{Ker}(s_I) = \mathrm{Ker}(s)$, consequently, $s = s_I$. Hence, M admits only one extremal state, $M = \bigcup_{t\in[0,1]_{\mathbb{H}}} M_t$, and $M_t = I_t$, where $I_t = s^{-1}(\{t\})$, for $t \in [0,1]_{\mathbb{H}}$, as stated.

Claim 3. If $a \in I_v$ and $b \in I_t$ for $v < t$, $v, t \in [0,1]_{\mathbb{H}}$, then $a < b$.

Let $\hat{s}_I$ denote the (unique) extension of s onto the ℓ-group (G,u), that is, s_I is a real-valued additive (in our case preserving multiplication) mapping on (G,u) preserving the order on G, and $s_I(u) = 1$.

There are two cases: (1') $Cg = Cg(a \wedge b)$ and (2') $Cg \neq Cg(a \wedge b)$.

(1') If $Cg = Cg(a \wedge b)$, then $a \wedge b \in g^{-1}Cg$, and while $g^{-1}Cg$ is prime, $a \in g^{-1}Cg$ or $b \in g^{-1}Cg$. Then $a = g^{-1}cg$ that gives $v = s_I(a) = \hat{s}_I(g^{-1}) + \hat{s}_I(c) + \hat{s}_I(g) = 0$ which is a contradiction. Similarly, $b \in g^{-1}Cg$ gives the same contradiction. Therefore (2') holds only.

(2') Transitivity guarantees the existence of an $h \in G$ such that $Cgh = Cg(a \wedge b)$. Hence, $Cgh = Cghh^{-1}(a \wedge b)$ which yields $h^{-1}(a \wedge b) \in (gh)^{-1}Cgh$. Since $h = g^{-1}cg(a \wedge b)$ we have $\hat{s}_I(h) = \hat{s}_I(g^{-1}) + \hat{s}_I(c) + \hat{s}_I(g) + \hat{s}_I(a \wedge b) = \hat{s}_I(a \wedge b) = s(a)$. Therefore, $h^{-1}(a \wedge b) = (h^{-1}a) \wedge (h^{-1}b) \in (gh)^{-1}C(gh)$. Since $(gh)^{-1}C(gh)$ is prime, and $h \in \hat{I}$, we get $h^{-1}a \in (gh)^{-1}C(gh)$. Then $h^{-1}a = (gh)^{-1}cgh$ for some $c \in C$, and $ga = ghh^{-1}a = cgh$, i.e., $Cga = Cgh$. But $Cga = Cgh = Cg(a \wedge b) \leq Cgb$.

Combining (1')–(2'), we have $Cga \leq Cgb$ for any $g \in G$, consequently, $a \leq b$, which yields $a < b$.

Finally, using Claim 1 and Claim 3, we have $I_0 \leqslant I_v \leqslant I_t \leqslant I_1$, for $v < t$, $v, t \in [0,1]_{\mathbb{H}} \setminus \{0,1\}$, which proves $M = (M_t : t \in [0,1]_{\mathbb{H}})$ and M is $\mathbb{H}$-perfect. By (iv) of Theorem 3.2, we have that M has a unique state.

(II) The general case $M/I \cong \Gamma(\mathbb{K}, 1)$, where $\mathbb{K}$ is a subgroup of $\mathbb{H}$, follows the same ideas as that for $\mathbb{K} = \mathbb{H}$ proving M is $\mathbb{K}$-perfect. □

4. Strong $\mathbb{H}$-perfect PMV-algebras and Their Representation

In this section, we introduce a stronger notion of $\mathbb{H}$-perfect PMV-algebras, called strong $\mathbb{H}$-perfect PMV-algebras, and we show when it can be represented in the form $\Gamma(\mathbb{H} \overrightarrow{\times} G, (1,0))$ for some unital ℓ-group G.

We say that a PMV-algebra M enjoys *unique extraction of roots of* 1 if $a, b \in M$ and na, nb exist in M, and $na = 1 = nb$, then $a = b$. Then every PMV-algebra $\Gamma(\mathbb{H} \overrightarrow{\times} G, (1,0))$ enjoys unique extraction of roots of 1 for any $n \geq 1$ and for any ℓ-group G. Indeed, let $k(s,g) = (1,0) = k(t,h)$ for some $s, t \in [0,1]_{\mathbb{H}}$, $g, h \in G$, $k \geq 1$. Then $ks = 1 = kt$ which yields $s = t > 0$, and $kg = 0 = kh$ implies $g = 0 = h$.

The following notion of a cyclic element was defined for PMV-algebras in [Dvu5, Dvu6] and for pseudo effect algebras in [DXY].

Let $n \geq 1$ be an integer. An element a of a PMV-algebra M is said to be *cyclic of order* n or simply *cyclic* if na exists in M and $na = 1$. If a is a cyclic element of order n, then $a^- = a^\sim$, indeed, $a^- = (n-1)a = a^\sim$. It is clear that 1 is a cyclic element of order 1.

Let $M = \Gamma(G,u)$ for some unital ℓ-group (G,u). An element $c \in M$ such that (a) $nc = u$ for some integer $n \geq 1$, and (b) $c \in C(H)$, where $C(H)$ is a commutative center of H, is said to be a *strong cyclic element of order* n.

For example, the PMV-algebra $M := \Gamma(\mathbb{Q} \overrightarrow{\times} G, (1,0))$, for every integer $n \geq 1$, M has a unique cyclic element of order n, namely $a_n = (\frac{1}{n}, 0)$. The PMV-algebra $\Gamma(\frac{1}{n}\mathbb{Z}, (1,0))$ for a prime number $n \geq 1$, has the only cyclic element of order n, namely $(\frac{1}{n}, 0)$. If $M = \Gamma(G,u)$ and G is a representable ℓ-group, G enjoys unique extraction of roots of 1, therefore, M has at most one cyclic element of order n.

In general, a PMV-algebra M can have two different cyclic elements of the same order. But if M has a strong cyclic element of order n, then it has a unique strong cyclic element of order n and a unique cyclic element of order n, [DvKo, Lem 5.2].

The following notions were introduced in [DvKo] for pseudo effect algebras.

We say that an $\mathbb{H}$-decomposition $(M_t : t \in [0,1]_{\mathbb{H}})$ of M has the *cyclic property* if there is a system of elements $(c_t \in M : t \in [0,1]_{\mathbb{H}})$ such that (i) $c_t \in M_t$ for any $t \in [0,1]_{\mathbb{H}}$, (ii) if $v+t \le 1$, $v,t \in [0,1]_{\mathbb{H}}$, then $c_v + c_t = c_{v+t}$, and (iii) $c_1 = 1$. Properties: (a) $c_0 = 0$; indeed, by (ii) we have $c_0 + c_0 = c_0$, so that $c_0 = 0$. (b) If $t = 1/n$, then $c_{\frac{1}{n}}$ is a cyclic element of order n.

Let $M = \Gamma(G,u)$, where (G,u) is a unital ℓ-group. An $\mathbb{H}$-decomposition $(M_t : t \in [0,1]_{\mathbb{H}})$ of M has the *strong cyclic property* if there is a system of elements $(c_t \in M : t \in [0,1]_{\mathbb{H}})$ such that (i) $c_t \in M_t \cap C(G)$ for any $t \in [0,1]_{\mathbb{H}}$, (ii) if $v+t \le 1$, $v,t \in [0,1]_{\mathbb{H}}$, then $c_v + c_t = c_{v+t}$, and (iii) $c_1 = 1$. We recall that if $t = 1/n$, $c_{\frac{1}{n}}$ is a strong cyclic element of order n.

For example, let $M = \Gamma(\mathbb{H} \overrightarrow{\times} G, (1,0))$, where G is an ℓ-group, and $M_t = \{(t,g) : (t,g) \in M\}$ for $t \in [0,1]_{\mathbb{H}}$. If we set $c_t = (t,0)$, $t \in [0,1]_{\mathbb{H}}$, then the system $(c_t : t \in [0,1]_{\mathbb{H}})$ satisfies (i)—(iii) of the strong cyclic property, and $(M_t : t \in [0,1]_{\mathbb{H}})$ is an $\mathbb{H}$-decomposition of M with the strong cyclic property.

Finally, we say that a PMV-algebra M has the $\mathbb{H}$-*strong cyclic property* if there is an $\mathbb{H}$-decomposition $(M_t : t \in [0,1]_{\mathbb{H}})$ of M with the strong cyclic property.

If $\mathbb{H} = \mathbb{Q}$, we can show an equivalent definition for the $\mathbb{Q}$-strong cyclic property, see also [DvKo, Prop 7.1]. Namely, we say that a PMV-algebra $M = \Gamma(G,u)$, where (G,u) is a unital ℓ-group, enjoys the *strong* 1-*divisibility property* if, given integer $n \ge 1$, there is an element $a_n \in C(G) \cap M$ such that $na_n = 1$. We see that a_n is a strong cyclic element of order n which is unique, and we denote it by $a_n = \frac{1}{n}1$. For any integer m, $0 \le m \le n$, we write $m\frac{1}{n}1 =: \frac{m}{n}1$.

Proposition 4.1. (1) *A PMV-algebra $M = \Gamma(G,u)$, where (G,u) is a unital ℓ-group, has the $\mathbb{Q}$-strong cyclic property if and only if M has the strong* 1-*divisibility property.*

(2) *A PMV-algebra $M = \Gamma(G,u)$, where (G,u) is a unital ℓ-group, has the $\frac{1}{n}\mathbb{Z}$-strong cyclic property if and only if M has a strong cyclic element of order n.*

Proof. (1) It follows from [DvKo, Prop 7.1].

(2) It follows from the definition of a strong cyclic element. □

Now we introduce a stronger notion of $\mathbb{H}$-perfect PMV-algebras which is inspired by an analogous one for PEAs' see [DvKo]. We say that a PMV-algebra M is *strong* $\mathbb{H}$-*perfect* if M possesses an $\mathbb{H}$-decomposition of M having the strong cyclic property.

A prototypical example of a strong $\mathbb{H}$-perfect PMV-algebra is the following.

Proposition 4.2. *Let G be an ℓ-group. Then the PMV-algebra*

$$\mathcal{M}_{\mathbb{H}}(G) := \Gamma(\mathbb{H} \overrightarrow{\times} G, (1,0)) \tag{4.1}$$

is a strong $\mathbb{H}$-perfect PMV-algebra.

We present a representation theorem for strong $\mathbb{H}$-perfect PMV-algebras by (4.1).

Theorem 4.3. *Let M be a strong $\mathbb{H}$-perfect PMV-algebra. Then there is a unique (up to isomorphism) ℓ-group G such that $M \cong \Gamma(\mathbb{H} \overrightarrow{\times} G, (1,0))$.*

Proof. Since M is a PMV-algebra, due to [Dvu2, Thm 3.9], there is a unique unital (up to isomorphism of unital ℓ-groups) ℓ-group (H,u) such that $M=\Gamma(H,u)$. Assume $(M_t : t\in[0,1]_{\mathbb{H}})$ is an $\mathbb{H}$-decomposition of M with the strong cyclic property and with a given system of elements $(c_t\in M : t\in[0,1]_{\mathbb{H}})$; due to Theorem 3.2, $(M_t : t\in[0,1]_{\mathbb{H}})$ is unique.

By (v)–(vi) of Theorem 3.2, $M_0=\mathrm{Infinit}(M)$ is an associative cancellative semigroup satisfying conditions of Birkhoff [Bir, Thm XIV.2.1], [Fuc, Thm II.4] which guarantees that M_0 is a positive cone of a unique (up to isomorphism) directed po-group G. Since M_0 is a lattice, we have that G is an ℓ-group.

By Theorem 3.2(iv), there is a unique $\mathbb{H}$-valued state s. This state is extremal, therefore, by [Dvu1, Prop 4.7], $s(a\wedge b)=\min\{s(a),s(b)\}$ for all $a,b\in M$, and the same is true for its extension $\hat{s}$ onto (H,u) and all $a,b\in H$.

Take the $\mathbb{H}$-strong cyclic PMV-algebra $\mathcal{M}_{\mathbb{H}}(G)$ defined by (4.1), and define a mapping $\phi : M\to\mathcal{M}_{\mathbb{H}}(G)$ by

$$\phi(x):=(t,x-c_t) \tag{4.2}$$

whenever $x\in M_t$ for some $t\in[0,1]_{\mathbb{H}}$, where $x-c_t$ denotes the difference taken in the group H.

Claim 1: ϕ is a well-defined mapping.

Indeed, M_0 is in fact the positive cone of an ℓ-group G which is a subgroup of H. Let $x\in M_t$. For the element $x-c_t\in H$, we define $(x-c_t)^+ := (x-c_t)\vee 0 = (x\vee c_t)-c_t\in M_0$ while $s((x\vee c_t)-c_t)=s(x\vee c_t)-s(c_t)=t-t=0$ and similarly $(x-c_t)^- := -((x-c_t)\wedge 0)=c_t-(x\wedge c_t)\in M_0$. This implies that $x-c_t=(x-c_t)^+-(x-c_t)^-\in G$.

Claim 2: The mapping ϕ is an injective and surjective homomorphism of pseudo effect algebras.

We have $\phi(0)=(0,0)$ and $\phi(1)=(1,0)$. Let $x\in M_t$. Then $x^-\in M_{1-t}$, and $\phi(x^-)=(1-t,x-c_{1-t})=(1,0)-(t,x-c_t)=\phi(x)^-$. In an analogous way, $\phi(x^\sim)=\phi(x)^\sim$.

Now let $x,y\in M$ and let $x+y$ be defined in M. Then $x\in M_{t_1}$ and $y\in M_{t_2}$. Since $x\le y^-$, we have $t_1\le 1-t_2$ so that $\phi(x)\le\phi(y^-)=\phi(y)^-$ which means $\phi(x)+\phi(y)$ is defined in $\mathcal{M}_{\mathbb{H}}(G)$. Then $\phi(x+y)=(t_1+t_2,x+y-c_{t_1+t_2})=(t_1+t_2,x+y-(c_{t_1}+c_{t_2}))=(t_1,x-c_{t_1})+(t_2,y-c_{t_2})=\phi(x)+\phi(y)$.

Assume $\phi(x)\le\phi(y)$ for some $x\in M_t$ and $y\in M_v$. Then $(t,x-c_t)\le(v,y-c_v)$. If $t=v$, then $x-c_t\le y-c_t$ so that $x\le y$. If $i<j$, then $x\in M_t$ and $y\in M_v$ so that $x<y$. Therefore, ϕ is injective.

To prove that ϕ is surjective, assume two cases: (i) Take $g\in G^+=M_0$. Then $\phi(g)=(0,g)$. In addition $g^-\in M_1$ so that $\phi(g^-)=\phi(g)^-=(0,g)^-=(1,0)-(0,g)=(1,-g)$. (ii) Let $g\in G$ and t with $0<t<1$ be given. Then $g=g_1-g_2$, where $g_1,g_2\in G^+=M_0$. Since $c_t\in M_t$, g_1+c_t exists in M and it belongs to M_t, and $g_2\le g_1+c_t$ which yields $(g_1+c_t)-g_2=(g_1+c_t)\setminus g_2\in M_t$. Hence, $g+c_t=(g_1+c_t)\setminus g_2\in M_t$ which entails $\phi(g+c_t)=(t,g)$.

Claim 3: If $x\le y$, then $\phi(y\setminus x)=\phi(y)\setminus\phi(x)$ and $\phi(x\,/\,y)=\phi(x)\,/\,\phi(y)$.

It follows from the fact that ϕ is a homomorphism of PEAs.

Claim 4: $\phi(x\wedge y)=\phi(x)\wedge\phi(y)$ and $\phi(x\vee y)=\phi(x)\vee\phi(y)$.

We have, $\phi(x), \phi(y) \geq \phi(x \wedge y)$. If $\phi(x), \phi(y) \geq \phi(w)$ for some $w \in M$, we have $x, y \geq w$ and $x \wedge y \geq w$. In the same way we deal with $\vee$.

Claim 5: ϕ is a homomorphism of PMV-algebras.

It is necessary to show that $\phi(x \oplus y) = \phi(x) \oplus \phi(y)$. It follows from the above claims and equality (2.2).

Consequently, M is isomorphic to $\mathcal{M}_{\mathbb{H}}(G)$ as PMV-algebras.

If $M \cong \Gamma(\mathbb{H} \overrightarrow{\times} G', (1,0))$, then G and G' are isomorphic ℓ-groups in view of the categorical equivalence, see [Dvu2, Thm 6.4] or Theorem 2.1. $\square$

5. Categorical Equivalence of Strong $\mathbb{H}$-perfect PMV-algebras

The categorical equivalence of strong n-perfect PMV-algebras with the category of ℓ-group was established in [Dvu5, Thm 7.7]. In this section, we generalize this result for the category of strong $\mathbb{H}$-perfect PMV-algebras. Our methods are similar to those used in [Dvu5].

Let $\mathcal{SPPMV}_{\mathbb{H}}$ be the category of strong $\mathbb{H}$-perfect pseudo MV-algebras whose objects are strong $\mathbb{H}$-perfect pseudo MV-algebras and morphisms are homomorphisms of PMV-algebras. Now let $\mathcal{G}$ be the category whose objects are ℓ-groups and morphisms are homomorphisms of unital ℓ-groups.

Define a mapping $\mathcal{M}_{\mathbb{H}} : \mathcal{G} \to \mathcal{SPPMV}_{\mathbb{H}}$ as follows: for $G \in \mathcal{G}$, let

$$\mathcal{M}_{\mathbb{H}}(G) := \Gamma(\mathbb{H} \overrightarrow{\times} G, (1,0))$$

and if $h : G \to G_1$ is an ℓ-group homomorphism, then

$$\mathcal{M}_{\mathbb{H}}(h)(t,g) = (t, h(g)), \quad (t,g) \in \Gamma(\mathbb{H} \overrightarrow{\times} G, (1,0)).$$

It is easy to see that $\mathcal{M}_{\mathbb{H}}$ is a functor.

Proposition 5.1. *$\mathcal{M}_{\mathbb{H}}$ is a faithful and full functor from the category $\mathcal{G}$ of ℓ-groups into the category $\mathcal{SPPMV}_{\mathbb{H}}$ of strong $\mathbb{H}$-perfect PMV-algebras.*

Proof. Let h_1 and h_2 be two morphisms from G into G' such that $\mathcal{M}_{\mathbb{H}}(h_1) = \mathcal{M}_{\mathbb{H}}(h_2)$. Then $(0, h_1(g)) = (0, h_2(g))$ for any $g \in G^+$, consequently $h_1 = h_2$.

To prove that $\mathcal{M}_{\mathbb{H}}$ is a full functor, suppose that f is a morphism from a strong $\mathbb{H}$-perfect PMV-algebra $\Gamma(\mathbb{H} \overrightarrow{\times} G, (1,0))$ into another one $\Gamma(\mathbb{H} \overrightarrow{\times} G_1, (1,0))$. Then $f(0,g) = (0,g')$ for a unique $g' \in G'^+$. Define a mapping $h : G^+ \to G'^+$ by $h(g) = g'$ iff $f(0,g) = (0,g')$. Then $h(g_1+g_2) = h(g_1)+h(g_2)$ if $g_1, g_2 \in G^+$. Assume now that $g \in G$ is arbitrary. Then $g = g_1 - g_2 = g_1' - g_2'$, where $g_1, g_2, g_1', g_2' \in G^+$, which gives $g_1 + g_2' = g_1' + g_2$, i.e., $h(g) = h(g_1) - h(g_2)$ is a well-defined extension of h from G^+ onto G.

Let $0 \leq g_1 \leq g_2$. Then $(0, g_1) \leq (0, g_2)$, which means h is a mapping preserving the partial order.

We have yet to show that h preserves $\wedge$ in G, i.e., $h(a \wedge b) = h(a) \wedge h(b)$ whenever $a, b \in G$. Let $a = a^+ - a^-$ and $b = b^+ - b^-$, and $a = -a^- + a^+$, $b = -b^- + b^+$. Since , $h((a^+ + b^-) \wedge (a^- + b^+)) = h(a^+ + b^-) \wedge h(a^- + b^+)$. Subtracting $h(b^-)$ from the right hand and $h(a^-)$ from the left hand, we obtain the statement in question.

Finally, we have established that h is a homomorphism of ℓ-groups, and $\mathcal{M}_{\mathbb{H}}(h) = f$ as claimed. $\square$

We recall that by a *universal group* for a PMV-algebra M we mean a pair (G, γ) consisting of an ℓ-group G and a G-valued measure $\gamma :\ M \to G^+$ (i.e., $\gamma(a + b) =$

$\gamma(a)+\gamma(b)$ whenever $a+b$ is defined in M) such that the following conditions hold: (i) $\gamma(M)$ generates G. (ii) If H is a group and $\phi: M \to H$ is an H-valued measure, then there is a group homomorphism $\phi^*: G \to H$ such that $\phi = \phi^* \circ \gamma$.

Due to [Dvu2], every PMV-algebra admits a universal group, which is unique up to isomorphism, and ϕ^* is unique. The universal group for $M = \Gamma(G,u)$ is (G, id) where id is the embedding of M into G.

Let $\mathcal{A}$ and $\mathcal{B}$ be two categories and let $f: \mathcal{A} \to \mathcal{B}$ be a morphism. Suppose that g, h be two morphisms from $\mathcal{B}$ to $\mathcal{A}$ such that $g \circ f = id_{\mathcal{A}}$ and $f \circ h = id_{\mathcal{B}}$, then g is a *left-adjoint* of f and h is a *right-adjoint* of f.

Proposition 5.2. *The functor $\mathcal{M}_{\mathbb{H}}$ from the category $\mathcal{G}$ into the category $\mathcal{SPPMV}_{\mathbb{H}}$ has a left-adjoint.*

Proof. We show, for a strong $\mathbb{H}$-perfect PMV-algebra M with an $\mathbb{H}$-decomposition $(M_t : t \in [0,1]_{\mathbb{H}})$ and a system $(c_t : t \in [0,1]_{\mathbb{H}})$ of elements of M satisfying (i)–(iii) of the strong cyclic property, there is a universal arrow (G, f), i.e., G is an object in $\mathcal{G}$ and f is a homomorphism from the PMV-algebra M into $\mathcal{M}_{\mathbb{H}}(G)$ such that if G' is an object from $\mathcal{G}$ and f' is a homomorphism from M into $\mathcal{M}_{\mathbb{H}}(G')$, then there exists a unique morphism $f^*: G \to G'$ such that $\mathcal{M}_{\mathbb{H}}(f^*) \circ f = f'$.

By Theorem 4.3, there is a unique (up to isomorphism of ℓ-groups) ℓ-group G such that $M \cong \Gamma(\mathbb{H} \overrightarrow{\times} G, (1,0))$. By [Dvu2, Thm 5.3], $(\mathbb{H} \overrightarrow{\times} G, \gamma)$ is a universal group for M, where $\gamma: M \to \Gamma(\mathbb{H} \overrightarrow{\times} G, (1,0))$ is defined by $\gamma(a) = (t, a - c_t)$, if $a \in M_t$. □

Define a mapping $\mathcal{P}_{\mathbb{H}} : \mathcal{SPPMV}_{\mathbb{H}} \to \mathcal{G}$ via $\mathcal{P}_{\mathbb{H}}(M) := G$ whenever $(\mathbb{H} \overrightarrow{\times} G, f)$ is a universal group for M. It is clear that if f_0 is a morphism from the PMV-algebra M into another one N, then f_0 can be uniquely extended to an ℓ-group homomorphism $\mathcal{P}_{\mathbb{H}}(f_0)$ from G into G_1, where $(\mathbb{H} \overrightarrow{\times} G_1, f_1)$ is a universal group for the strong $\mathbb{H}$-perfect PMV-algebra N.

Proposition 5.3. *The mapping $\mathcal{P}_{\mathbb{H}}$ is a functor from the category $\mathcal{SPPMV}_{\mathbb{H}}$ into the category $\mathcal{G}$ which is a left-adjoint of the functor $\mathcal{M}_{\mathbb{H}}$.*

Proof. It follows from the properties of the universal group. □

Now we present the main result on a categorical equivalence of the category of strong $\mathbb{H}$-perfect PMV-algebras and the category of $\mathcal{G}$.

Theorem 5.4. *The functor $\mathcal{M}_{\mathbb{H}}$ defines a categorical equivalence of the category $\mathcal{G}$ and the category $\mathcal{SPPMV}_{\mathbb{H}}$ of strong $\mathbb{H}$-perfect PMV-algebras.*

In addition, suppose that $h: \mathcal{M}_{\mathbb{H}}(G) \to \mathcal{M}_{\mathbb{H}}(H)$ is a homomorphism of pseudo effect algebras, then there is a unique homomorphism $f: G \to H$ of unital po-groups such that $h = \mathcal{M}_{\mathbb{H}}(f)$, and

(i) *if h is surjective, so is f;*
(ii) *if h is injective, so is f.*

Proof. According to [MaL, Thm IV.4.1], it is necessary to show that, for a strong $\mathbb{H}$-perfect PMV-algebra M, there is an object G in $\mathcal{G}$ such that $\mathcal{M}_{\mathbb{H}}(G)$ is isomorphic to M. To show that, we take a universal group $(\mathbb{H} \overrightarrow{\times} G, f)$. Then $\mathcal{M}_{\mathbb{H}}(G)$ and M are isomorphic. □

Theorem 5.4 entails directly the following statement.

Corollary 5.5. *If $\mathbb{H}_1$ and $\mathbb{H}_2$ are two subgroups of $\mathbb{R}$ containing the number 1, then the categories $\mathcal{SPPMV}_{\mathbb{H}_1}$, $\mathcal{SPPMV}_{\mathbb{H}_2}$ and the category $\mathcal{G}$ of ℓ-groups are mutually categorically equivalent.*

Theorem 5.6. *Let G be a doubly transitive ℓ-group. Then $\mathcal{V}(\mathcal{SPPMV}_{\mathbb{H}}) = \mathcal{V}(\mathcal{M}_{\mathbb{H}}(G))$.*

In particular, an identity holds in every strong $\mathbb{H}$-perfect PMV-algebra if and only if it holds in $\mathcal{M}_{\mathbb{H}}(G)$.

Proof. Let G be a doubly transitive ℓ-group, and define a strong $\mathbb{H}$-perfect PMV $\mathcal{M}_{\mathbb{H}}(G)$ by (4.1).

Let M be a strong $\mathbb{H}$-perfect PMV-algebra. Due to Theorem 4.3, there is a unique (up to isomorphism of unital ℓ-groups) ℓ-group G_M such that $M = \mathcal{M}_{\mathbb{H}}(G_M)$. Since every doubly transitive ℓ-group generates the variety $\mathcal{G}$ of ℓ-groups, [Gla, Lem. 10.3.1], there exist a homomorphism f of ℓ-groups and an ℓ-group K such that $f(K) = G_M$ and $K \subseteq G^J$, where J is an index set. Due to Theorem 5.4, $M = \mathcal{M}_{\mathbb{H}}(G_M) = \mathcal{M}_{\mathbb{H}}(f)(\mathcal{M}_{\mathbb{H}}(K))$.

Define a map $\rho: \mathcal{M}_{\mathbb{H}}(G^J) \to (\mathcal{M}_{\mathbb{H}}(G))^J$ via $\rho(0,(g_j)_{j\in J}) = \{(0,g_j)\}_{j\in J}$ and $\rho(1,(-g_j)_{j\in J}) = \{(1,-g_j)\}_{j\in J}$ for $g_j \in G^+$, and $\rho(t,g_j) = \{(t,g_j)\}_{j\in J}$, $t \in [0,1]_{\mathbb{H}} \setminus \{0,1\}$, $g_j \in G$ for $j \in J$. Then ρ is an embedding, and $\mathcal{M}_{\mathbb{H}}(G^J) \in \mathcal{V}(\mathcal{M}_{\mathbb{H}}(G))$. Since $\mathcal{M}_{\mathbb{H}}(K)$ is a subalgebra of $\mathcal{M}_{\mathbb{H}}(G^J)$, we have $\mathcal{M}_{\mathbb{H}}(K) \in \mathcal{V}(\mathcal{M}_{\mathbb{H}}(G))$ and $M \in \mathcal{V}(\mathcal{M}_{\mathbb{H}}(G))$ because it is a homomorphic image of $\mathcal{M}_{\mathbb{H}}(K) \in \mathcal{V}(\mathcal{M}_{\mathbb{H}}(G))$. $\square$

An example of a doubly transitive permutation ℓ-group is the system of all automorphisms, $\mathrm{Aut}(\mathbb{R})$, of the real line $\mathbb{R}$, or the next example:

Let $u \in \mathrm{Aut}(\mathbb{R})$ be the translation $tu = t+1$, $t \in \mathbb{R}$, and

$$\mathrm{BAut}(\mathbb{R}) = \{g \in \mathrm{Aut}(\mathbb{R}) : \ \exists\, n \in \mathbb{N},\ \ u^{-n} \le g \le u^n\}.$$

Then $(\mathrm{BAut}(\mathbb{R}), u)$ is a doubly transitive unital ℓ-permutation group, and according to [DvHo, Cor. 4.9], the variety of PMV-algebras generated by $\Gamma(\mathrm{BAut}(\mathbb{R}), u)$ is the variety of all PMV-algebras.

6. Weak $\mathbb{H}$-perfect PMV-algebras

In this section, we introduce another family of $\mathbb{H}$-perfect PMV-algebras, called weak $\mathbb{H}$-perfect PMV-algebras. They can be represented in the form $\Gamma(\mathbb{H} \overrightarrow{\times} G, (1,b))$, where G is an ℓ-group and $0 < b \in G^+$. Such PVM-algebras were studied in [Dvu5] for the case when $\mathbb{H}$ is a cyclic subgroup of $\mathbb{R}$.

We say that an $\mathbb{H}$-perfect pseudo MV-algebra $M = (M_t : t \in [0,1]_{\mathbb{H}})$, where $M = \Gamma(G,u)$, is *weak* if there is a system $(c_t : t \in [0,1]_{\mathbb{H}})$ of elements of M such that (i) $c_0 = 0$, (ii) $c_t \in C(G) \cap M_t$, for any $t \in [0,1]_{\mathbb{H}}$, and (iii) $c_{v+t} = c_v + c_t$ whenever $v + t \le 1$. We note that in contrast to strong cyclic property, we do not assume $c_1 = 1$. In addition, a weak $\mathbb{H}$-perfect PMV-algebra M is strong iff $c_1 = 1$.

Whereas every strong $\mathbb{H}$-perfect PMV-algebra is symmetric, for weak $\mathbb{H}$-perfect PMV-algebras this is not necessarily a case.

For example, if g_0 is a positive element of an ℓ-group G such that $g_0 \notin C(G)$, then $M = \Gamma(\mathbb{H} \overrightarrow{\times} G, (1,g_0))$ is a weak $\mathbb{H}$-perfect PMV-algebra which is neither symmetric, nor strong; we set $c_t = (t,0)$ for any $t \in [0,1]_{\mathbb{H}}$. Then $c_1 = (1,0) < (1,g_0)$.

Theorem 6.1. *Let $M = (M_t : t \in [0,1]_{\mathbb{H}})$ be a weak $\mathbb{H}$-perfect PMV-algebra which is not strong. Then there is a unique (up to isomorphism) ℓ-group G with an element $b \in G^+$, $b > 0$, such that $M \cong \Gamma(\mathbb{Z} \overrightarrow{\times} G, (n,b))$.*

Proof. Assume $M = \Gamma(H,u)$ for some unital ℓ-group (H,u). As in the proof of Theorem 4.3, we can found a unique (up to isomorphism) ℓ-group G such that $\mathrm{Infinit}(M) = M_0$ is the positive cone of G, moreover, G is an ℓ-subgroup of H. We recall that if s is a unique state on M, it can be extended to a unique state, $\hat{s}$, on the unital ℓ-group (G,u). Since by (iv) Theorem 3.2, $M_0 = \mathrm{Ker}(s)$, we have $G = \mathrm{Ker}(\hat{s})$.

Since M is not strong, then $c_1 < 1 =: u$. Set $b = u \setminus c_1 = 1 - c_1 \in M_0 \setminus \{0\}$, and define a mapping $h: \ M \to \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1,b))$ as follows

$$\phi(x) = (t, x - c_t) \tag{6.1}$$

whenever $x \in M_t$; we note that the subtraction $x - c_t$ is defined in the ℓ-group H. In the same way as in (3.2), we can show that ϕ is a well-defined mapping.

We have (1) $\phi(0) = (0,0)$, (2) $\phi(1) = (1, 1-c_1) = (1,b)$, (3) $\phi(c_t) = (t,0)$, (4) $\phi(x^\sim) = (1-t, -x+u-c_{1-t}) = (1-t, -x+b+c_t)$, $\phi(x)^\sim = -\phi(x) + (1,b) = -(t, x-c_t) + (1,b) = (1-t, -x+b+c_t)$ and similarly (5) $\phi(x^-) = \phi(x)^-$.

Using the same steps as those used in the proofs of all claims of the proof of Theorem 4.3, we can prove that ϕ is an injective and surjective homomorphism of pseudo MV-algebras as was claimed. □

We note that Theorem 6.1 is a generalization of Theorem 4.3, because Theorem 4.3 in fact follows from Theorem 6.1 when we have $b = 0$.

Finally, let $\mathcal{WPPMV}_{\mathbb{H}}$ be the category of weak $\mathbb{H}$-perfect PMV-algebras whose objects are weak $\mathbb{H}$-perfect PMV-algebras and morphisms are homomorphisms of PMV-algebras. Similarly, let $\mathcal{L}_{\mathrm{b}}$ be the category whose objects are couples (G,b), where G is an ℓ-group and b is a fixed element from G^+, and morphisms are ℓ-homomorphisms of ℓ-groups preserving fixed elements b.

Define a mapping $\mathcal{F}_{\mathbb{H}}$ from the category $\mathcal{L}_{\mathrm{b}}$ into the category $\mathcal{WPPMV}_{\mathbb{H}}$ as follows:

Given $(G,b) \in \mathcal{L}_{\mathrm{b}}$, we set

$$\mathcal{F}_{\mathbb{H}}(G,b) := \Gamma(\mathbb{H} \overrightarrow{\times} G, (1,b)), \tag{6.2}$$

and if $h : (G,b) \to (G_1, b_1)$, then

$$\mathcal{F}_{\mathbb{F}}(h)(t,g) = (t, h(g)), \quad (t,g) \in \Gamma(\mathbb{H} \overrightarrow{\times} G, (1,b)).$$

It is easy to see that $\mathcal{F}_{\mathbb{H}}$ is a functor.

In the same way as the categorical equivalence of strong $\mathbb{H}$-perfect PMV-algebras was proved in Section 5, we can prove the following theorem.

Theorem 6.2. *The functor $\mathcal{F}_{\mathbb{H}}$ defines a categorical equivalence of the category $\mathcal{L}_{\mathrm{b}}$ and the category $\mathcal{WPPMV}_{\mathbb{H}}$ of weak $\mathbb{H}$-perfect PMV-algebras.*

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